

# IGUSA'S LOCAL ZETA FUNCTIONS OF SEMIQUASIHOMOGENEOUS POLYNOMIALS

W. A. ZÚÑIGA-GALINDO\*

ABSTRACT. In this paper, we prove the rationality of Igusa's local zeta functions of semi-quasihomogeneous polynomials with coefficients in a non-archimedean local field  $K$ . The proof of this result is based on Igusa's stationary phase formula and some ideas on Néron  $\pi$ -desingularization.

## 1. Introduction

Let  $K$  be a non-archimedean local field, and let  $\mathcal{O}_K$  be the ring of integers of  $K$ . Let  $\pi$  be a uniformizing parameter of  $K$ , and let the residue field of  $K$  be  $\mathbb{F}_q$  a finite field with  $q = p^r$  elements. Let  $v$  denote the valuation of  $K$  such that  $v(\pi) = 1$ . For  $x \in K$ , let  $|x|_K = q^{-v(x)}$ . Let  $|dx|$  be the Haar measure on  $K^n$  so normalized that the measure of  $\mathcal{O}_K^n$  is equal to one. Let  $f(x) \in K[x]$ ,  $x = (x_1, \dots, x_n)$ , the Igusa local zeta function associated to  $f$  is defined by

$$Z(f, s) = \int_{\mathcal{O}_K^n} |f(x)|_K^s |dx|, \quad s \in \mathbb{C}, \operatorname{Re}(s) > 0.$$

The local zeta function  $Z(f, s)$  is a holomorphic function on the semiplane  $\operatorname{Re}(s) > 0$ . In the case of  $K$  having characteristic zero, Igusa ([7], [8]) and Denef ([3]) proved that  $Z(f, s)$  is a rational function of  $q^{-s}$ . At the present time, the techniques used by Igusa (resolution of singularities) and Denef (elimination of quantifiers in  $\mathbb{Q}_p$ ) are not available in positive characteristic, so in this case the rationality of  $Z(f, s)$  is still an open problem. The local zeta function contains information about the number of solutions of the congruence  $f(x) \equiv 0 \pmod{\pi^j \mathcal{O}_K}$ , (see e.g. [4]). More precisely, if

$$N_j := \operatorname{Card}\{x \in (\mathcal{O}_K/\pi^j \mathcal{O}_K)^n \mid f(x) \equiv 0 \pmod{\pi^j \mathcal{O}_K}\},$$

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1991 *Mathematics Subject Classification*. Primary 11D79, 11S40, 14G10.

*Key words and phrases*. Local zeta functions, rationality in positive characteristic.

\* This work was supported by COLCIENCIAS, contract # 063-98

and  $P(t)$  is the Poincaré series  $P(t) = \sum_{j \geq 0} N_j(q^{-n}t)^j$ , then

$$Z(f, s) = P(q^{-s}) - q^s(P(q^{-s}) - 1).$$

In this paper, we shall study the local zeta functions of semiquasihomogeneous polynomials with an absolutely algebraically isolated singularity at the origin of  $K^n$ .

Let  $f(x)$  be a polynomial with coefficients in  $K$ , and  $V_f$  the corresponding  $K$ -hypersurface. We call a point  $P \in K^n$  an *absolutely algebraically isolated singularity* of  $V_f(K)$ , if the only solution of the system of equations

$$f(x) = \frac{\partial f}{\partial x_1}(x) = \dots = \frac{\partial f}{\partial x_n}(x) = 0,$$

over an algebraic fixed closure of  $K$ , is the point  $P$ .

Let  $\alpha_1, \dots, \alpha_n$  be  $n$  relatively prime and positive integers. A polynomial  $f(x) \in K[x]$  is called a *quasihomogeneous polynomial* of weight  $d$  and exponents  $\alpha_1, \dots, \alpha_n$ , if it satisfies:

$$f(t^{\alpha_1}x_1, \dots, t^{\alpha_n}x_n) = t^d f(x), \quad \text{for every } t \in K,$$

and the origin of  $K^n$  is an absolutely algebraically isolated singularity of  $K$ -hypersurface  $V_f$ .

A polynomial  $F(x)$  is called a *semiquasihomogeneous polynomial* if it has the form  $f(x) + \sum b_i e_i(x) \in K[x]$ , where  $f(x)$  is a quasihomogeneous polynomial, and each monomial  $e_i(x) = x_1^{m_1} \dots x_n^{m_n}$  satisfies  $\sum \alpha_i m_i > d$ , and the origin of  $K^n$  is an absolutely algebraically isolated singularity of  $K$ -hypersurface  $V_F$ . We call the polynomial  $f(x)$  the quasihomogeneous part of  $F(x)$ .

We put  $|\alpha| = \sum_i \alpha_i$ , for any  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ . We use the notation  $Z(f, D, s)$  for the integral  $\int_D |f(x)|_K^s |dx|$ . In the case of  $D = \mathcal{O}_K^n$ , we use the simplified notation  $Z(f, s)$ .

The main result of this paper is the following:

**Theorem 3.5.** *Let  $F(x) \in K[x]$  be a semiquasihomogeneous polynomial whose quasihomogeneous part  $f(x)$  has weight  $d$  and exponents  $\alpha_1, \dots, \alpha_n$ . Then Igusa's local zeta function of  $F(x)$  is a rational function of  $q^{-s}$ . More precisely,*

$$Z(F, s) = \frac{L(q^{-s})}{(1 - q^{-1}q^{-s})(1 - q^{-|\alpha|}q^{-ds})}, \quad (1.1)$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$ . Furthermore, the polynomial  $L(q^{-s})$  can be computed effectively.

If in addition the polynomial  $F$  is nondegenerate for its Newton diagram and if  $K$  has characteristic zero, then a very different way of calculating  $Z(F, s)$  is given in [5]. The proof of theorem 3.5 gives an effective method to compute the local zeta functions of semiquasihomogeneous polynomials.

We say that a singular point  $\bar{P} \in V_{\bar{f}}(\mathbb{F}_q)$ , where  $\bar{f}$  is the reduction modulo  $\pi$  of  $f$ , is a non-liftable singularity of the hypersurface  $V_f$ , if for every singular point  $Q \in V_f(K)$ ,  $Q \in \mathcal{O}_K^n$ , the reduction modulo  $\pi$  of  $Q$  is different of  $\bar{P}$ . The proof of theorem 3.5 shows that the numerator of the zeta function  $Z(F, s)$  depends on the non-liftable singularities of the closed fiber of the hypersurface  $V_F$ , and the denominator depends on the singularity of the generic fiber of  $V_F$ . More precisely, the denominator depends on Newton's diagram of  $F(x)$ . In the proof of theorem 3.5, we use Igusa's formula of stationary phase for  $\pi$ -adic integrals ([10]) and some ideas on Néron  $\pi$ -desingularization ([13], sect. 17, 18).

As a consequence of theorem 3.5, we obtain the following three corollaries.

**Corollary 3.6.** *Let  $K$  be a global field and  $F(x) \in K[x]$  be a semiquasihomogeneous polynomial whose quasihomogeneous part  $f(x)$  has weight  $d$  and exponents  $\alpha_1, \dots, \alpha_n$ . Then for every non-archimedean valuation  $v$  of  $K$ , Igusa's local zeta function of  $F(x)$  on the completion  $K_v$  of  $K$  is a rational function of form (1.1). If  $K$  is a number field and  $F(x)$  is non-degenerate for its Newton's diagram, then the real parts of the poles of the zeta function  $Z(F, s)$  are roots of the Bernstein polynomial of  $F(x)$ .*

For the definition of the Bernstein polynomial and its computation in the non-degenerate case see reference [2]. The last part of corollary 3.6 is a special case of a more general result due to Loeser (cf. [11], thm. 5.5.1).

**Corollary 3.7.** *Let  $K$  be a global field and  $\mathcal{O}_K$  its ring of integers. Let  $F(x) \in \mathcal{O}_K[x]$  be a semiquasihomogeneous polynomial whose quasihomogeneous part  $f(x)$  has weight  $d$  and exponents  $\alpha_1, \dots, \alpha_n$ . Then for every non-archimedean valuation  $v$  of  $K$ , the number of solutions  $N_j(F, v)$  of the congruence*

$$F(x) \equiv 0 \pmod{\pi^j \mathcal{O}_{K_v}},$$

where  $\mathcal{O}_{K_v}$  is the ring of integers of the completion  $K_v$ , satisfies

$$\limsup_{j \rightarrow \infty} N_j(F, v)^{1/j} \leq \begin{cases} q^{n-|\alpha|/d} & \text{if } |\alpha|/d \leq 1, \\ q^{n-1} & \text{if } |\alpha|/d > 1. \end{cases}$$

The zeta functions of plane curves, with only an absolutely analytically irreducible singularity at the origin, have been extensively studied when the characteristic of  $K$  is zero, by Igusa ([9]), Meuser ([12]), among others. Let  $f(x, y) \in K[x, y]$  be an absolutely analytically irreducible polynomial. Thus there exist  $(\alpha_1, \alpha_2) \in \mathbb{N}^2$ , relatively prime integers, and an integer  $d$ , such that  $f(x, y) = f_d(x, y) + g(x, y)$  and the monomials  $x^n y^m$  of  $f_d(x, y)$  and  $g(x, y)$  satisfy  $\alpha_1 n + \alpha_2 m = d$  and  $\alpha_1 n + \alpha_2 m > d$ , respectively. The origin of  $K^2$  is an absolutely algebraically isolated singularity of the plane curve  $V_f$ . If it is also valid for  $V_{f_d}$ , then  $f$  is a semiquasihomogeneous polynomial in the sense of the definition given for us. As a consequence of theorem 3.5, we obtain a precise description of the local zeta function associated with this type of polynomials.

**Corollary 3.8.**

*Let  $f(x, y) \in K[x, y]$  be an absolutely analytically irreducible polynomial, such that the origin of  $K^2$  is an absolutely algebraically isolated singularity of the plane curve  $V_{f_d}$ . Then the Igusa local zeta function  $Z(f, s)$  is a rational function of  $q^{-s}$  of form (1.1).*

## 2. Preliminaires

In [10] Igusa introduced the stationary phase formula for  $\pi$ -adic integrals and suggested that a closer examination of this formula might lead to a proof of the rationality of  $Z(f, s)$  in any characteristic. The above suggestion has been our main motivation for this paper. In this section we review Igusa's stationary phase formula and some ideas on Néron  $\pi$ -desingularization.

We denote by  $\bar{x}$  the image of an element of  $\mathcal{O}_K$  under the canonical homomorphism  $\mathcal{O}_K \longrightarrow \mathcal{O}_K/\pi\mathcal{O}_K \cong \mathbb{F}_q$ , i.e, the reduction modulo  $\pi$ . Given  $f(x) \in \mathcal{O}_K[x]$  such that not all its coefficients are in  $\pi\mathcal{O}_K$ , we denote by  $\overline{f(x)}$  the polynomial obtained by reducing modulo  $\pi$  the coefficients of  $f(x)$ .

For any commutative ring  $A$  and  $f(x) \in A[x]$ , we denote by  $V_f(A)$ , the set of  $A$ -valued points of the hypersurface  $V_f$  defined by  $f$ , and by  $Sing_f(A)$ , the set of  $A$ -valued singular points of  $V_f$ , i.e.,

$$Sing_f(A) = \{x \in A^n \mid f(x) = \frac{\partial f}{\partial x_1}(x) = \dots = \frac{\partial f}{\partial x_n}(x) = 0\}.$$

We fix a lifting  $R$  of  $\mathbb{F}_q$  in  $\mathcal{O}_K$ . Thus, the set  $R$  is mapped bijectively onto  $\mathbb{F}_q$  by the canonical homomorphism  $\mathcal{O}_K \longrightarrow \mathcal{O}_K/\pi\mathcal{O}_K$ .

Let  $f(x) \in \mathcal{O}_K[x]$  be a polynomial in  $n$  variables,  $P_1 = (y_1, \dots, y_n) \in \mathcal{O}_K^n$ , and  $m_{P_1} = (m_1, \dots, m_n) \in \mathbb{N}^n$ . We call a  $K^n$ -isomorphism  $\psi_{m_{P_1}}(x)$  a *dilatation*, if it

satisfies  $\psi_{m_{P_1}}(x) = (z_1, \dots, z_n)$ ,  $z_i = y_i + \pi^{m_{P_1} x_i}$ , for each  $i = 1, 2, \dots, n$ . We define the *dilatation* of  $f(x)$  at  $P_1$  induced by  $\psi_{m_{P_1}}(x)$ , as

$$f_{P_1}(x) := \pi^{-e_{P_1}} f(\psi_{m_{P_1}}(x)), \quad (2.1)$$

where  $e_{P_1}$  is the minimum order of  $\pi$  in the coefficients of  $f(\psi_{m_{P_1}}(x))$ . We call the  $K$ -hypersurface  $V_{f_{P_1}}$  the dilatation of  $V_f$  at  $P_1$  induced by  $\psi_{m_{P_1}}(x)$ , the number  $e_{P_1}$  the *arithmetic multiplicity of  $f(x)$  at  $P_1$  by  $\psi_{m_{P_1}}(x)$*  and the set  $S(f_{P_1})$ , the lifting of  $\text{Sing}_{\overline{f_{P_1}}}(\mathbb{F}_q)$ , the *first generation of descendants of  $P_1$* . By dilatating  $f_{P_1}$  at each  $P_2 \in S(f_{P_1})$ , by some  $\psi_{m_{P_2}}(x)$ , we obtain  $(f_{P_1, P_2}, S(f_{P_1, P_2}))_{P_2 \in S(f_{P_1})}$ . The union of the sets  $(S(f_{P_1, P_2}))_{P_2 \in S(f_{P_1})}$  is the second generation of descendants of  $P_1$ . Given a sequence of dilatations  $(\psi_{m_{P_k}}(x))_k$ , we define inductively  $e_{P_1, \dots, P_k}$  and  $f_{P_1, \dots, P_k}(x)$ ,  $S(f_{P_1, \dots, P_k})$  as follows:

$$f_{P_1, \dots, P_k}(x) = \begin{cases} f(x) & \text{if } k = 0, \\ \pi^{-e_{P_1, \dots, P_k}} f_{P_1, \dots, P_{k-1}}(\psi_{m_{P_k}}(x)) & \text{if } k \geq 1, \end{cases} \quad (2.2)$$

where  $P_k$  runs through the descendants of  $k-1$  generation of  $P_1$ . The union of the sets  $S(f_{P_1, \dots, P_k})$ , is called the  *$k$ -generation of descendants of  $P_1$* .

Along this paper we shall use several types of dilatations, i.e., dilatations with different  $m$ 's, however the specific value of  $m$  will be clear from the context. The dilatations were introduced by Néron (cf. [13], sect. 18). These transformations play an important role in the process of desingularization of the closed fiber of a scheme over a discrete valuation ring whose generic fiber is non-singular. A modern exposition of the Néron  $\pi$ -desingularization can be found in [1], sect. 4.

Now, we review Igusa's stationary phase formula, from the point of view of the dilatations. For that, we fix the  $m_{P_k}$ 's equal to  $(1, \dots, 1) \in \mathbb{N}^n$  in (2.2).

Let  $\overline{D}$  be a subset of  $\mathbb{F}_q^n$  and  $D$  its preimage under the canonical homomorphism  $\mathcal{O}_K \longrightarrow \mathcal{O}_K/\pi\mathcal{O}_K \cong \mathbb{F}_q$ . Let  $S(f, D)$  denote the subset of  $R^n$  (the set of representatives of  $\mathbb{F}_q^n$  in  $\mathcal{O}_K^n$ ) mapped bijectively to the set  $\text{Sing}_{\overline{f}}(\mathbb{F}_q) \cap \overline{D}$ . We use the simplified notation  $S(f)$ , in the case of  $D = \mathcal{O}_K^n$ . Also we define:

$$\nu(\overline{f}, D) := q^{-n} \text{Card}\{\overline{P} \in \overline{D} \mid \overline{P} \notin V_{\overline{f}}(\mathbb{F}_q)\},$$

$$\sigma(\overline{f}, D) := q^{-n} \text{Card}\{\overline{P} \in \overline{D} \mid \overline{P} \text{ is a non-singular point of } V_{\overline{f}}(\mathbb{F}_q)\}.$$

In order to simplify the notation, we shall use  $\nu(\bar{f}), \sigma(\bar{f})$  instead of  $\nu(\bar{f}, D), \sigma(\bar{f}, D)$ , respectively. The dependence of a particular set  $D$ , it will be clear from the context.

With all this, we are able to establish Igusa's stationary phase formula for  $\pi$ -adic integrals:

**Igusa's Stationary Phase Formula.** ([10], p. 17)

$$\int_D |f(x)|_K^s |dx| = \nu(\bar{f}) + \sigma(\bar{f}) \frac{(1 - q^{-1})q^{-s}}{(1 - q^{-1}q^{-s})} + \sum_{P \in S(f, D)} q^{-n - e_P s} \int_{\mathcal{O}_K^n} |f_P(x)|_K^s |dx|, \quad (2.3)$$

where  $Re(s) > 0$ . Formula (2.3) is obtained in the following form. Suppose that  $\bar{D} = \{\bar{P}_1, \dots, \bar{P}_N\}$  and let  $P_i$  be the lifting of  $\bar{P}_i$ . Then the set  $D$  is the disjoint union  $\bigcup_P D_P$ , where  $\bar{P} = (\bar{y}_1, \dots, \bar{y}_n) \in \bar{D}$  and  $D_P$  is defined as

$$D_P = \{(x_1, \dots, x_n) \in D \mid x_i = y_i + \pi z_i, z_i \in \mathcal{O}_K, i = 1, 2, \dots, n\}.$$

Thus

$$\int_D |f(x)|_K^s |dx| = \sum_{\bar{P}} \int_{D_P} |f(x)|_K^s |dx| = \sum_{\bar{P}} q^{-n - e_P s} \int_{\mathcal{O}_K^n} |f_P(x)|_K^s |dx|.$$

The integrals corresponding to the  $P$ 's for which  $\bar{P} \notin V_{\bar{f}}(\mathbb{F}_q)$  are easily computable. The integrals corresponding to the  $P$ 's for which  $\bar{P}$  is a non-singular point of  $V_{\bar{f}}(\mathbb{F}_q)$  are computed using the implicit function theorem (cf. [10], p. 177).

By iterating the stationary phase formula, we obtain the following expansion for  $Z(f, s)$  (cf. [10], p. 178):

$$\begin{aligned} Z(f, s) &= \sum_{k \geq 0} q^{-kn} \left( \sum_{P_1, P_2, \dots, P_k} \nu(\bar{f}_{P_1, \dots, P_k}) q^{-E(P_1, \dots, P_k)s} \right) \\ &+ \frac{(1 - q^{-1})q^{-s}}{(1 - q^{-1}q^{-s})} \sum_{k \geq 0} q^{-kn} \left( \sum_{P_1, P_2, \dots, P_k} \sigma(\bar{f}_{P_1, \dots, P_k}) q^{-E(P_1, \dots, P_k)s} \right) \end{aligned} \quad (2.4)$$

where  $E(P_1, \dots, P_k) := e_{P_1} + \dots + e_{P_1, \dots, P_k}$ . Expansion (2.4) converges absolutely on the semiplane  $Re(s) > 0$ .

Now, we summarize some ideas on Néron  $\pi$ -desingularization (see [13], sect. 17, 18) to be used in the next sections. Let  $f(x) \in \mathcal{O}_K[x]$  be a polynomial, and  $P \in V_f(\mathcal{O}_K)$ . Néron introduced the following measure of singularity at  $P$ :

$$l(f, P) := \inf_i \left( v\left(\frac{\partial f}{\partial x_i}(P)\right) \right).$$

The Jacobian criterion implies that  $P$  is a smooth point of  $V_f(K)$  if and only if  $l(f, P)$  is finite.  $\overline{P}$  is a smooth point of  $V_{\overline{f}}(\mathbb{F}_q)$  if and only if  $l(f, P) = 0$ . In this paper, we introduce the following measure of singularity at an integer point  $P$ , satisfying  $\overline{P} \in V_{\overline{f}}(\mathbb{F}_q)$ .

**Definition 2.1.** Let  $f(x) \in \mathcal{O}_K[x]$  be a polynomial and  $P \in \mathcal{O}_K^n$  a point, such that  $P \notin \text{Sing}_f(\mathcal{O}_K)$  and  $\overline{P} \in V_{\overline{f}}(\mathbb{F}_q)$ . We define

$$L(f, P) := \inf \left( v(f(P)), v\left(\frac{\partial f}{\partial x_1}(P)\right), \dots, v\left(\frac{\partial f}{\partial x_n}(P)\right) \right).$$

Let us observe that  $L(f, P) = 0$  if and only if

$$\overline{f(x)} = \alpha_0 + \sum_j \alpha_j (x_j - \overline{a_j}) + (\text{degree} \geq 2),$$

where  $P = (a_1, \dots, a_n)$ ,  $\alpha_0 \in \mathbb{F}_q^*$  or  $\alpha_j \in \mathbb{F}_q^*$  for some  $j = 1, 2, \dots, n$ . We also observe that  $l(f, p) = L(f, p)$  if  $P \in V_f(\mathcal{O}_K)$ . The integer  $L(f, P)$  has similar properties to those of  $l(f, P)$ . This integer appears naturally associated to Igusa's stationary phase, as we shall see later on.

We denote by  $A_r$ ,  $r = (r_1, \dots, r_n) \in (\mathbb{N} \setminus \{0\})^n$ , the set

$$A_r := \{x \in \mathcal{O}_K^n \mid v(x_i) \geq r_i, i = 1, \dots, n\}.$$

From a geometrical point of view,  $A_r$  is a polydisc in  $\mathcal{O}_K^n$  centered at the origin. The complement of  $A_r$  in  $\mathcal{O}_K^n$  is denoted as  $A_r^c$ .

The following proposition is a simple reformulation of proposition 17 in sect. 17 of [13]. However for our convenience, we prove it below.

**Proposition 2.2.** (Néron, [13], sect. 17, prop. 17) Let  $f(x) \in \mathcal{O}_K[x]$  be a polynomial,  $P \in \mathcal{O}_K^n$  an absolutely algebraically isolated singularity of the hypersurface  $V_f$ , and let  $D \subseteq \mathcal{O}_K^n$  be a subset such that  $D \cap (P + A_r) = \emptyset$ , for some  $r \in (\mathbb{N} \setminus \{0\})^n$ . Then

$$L(f, Q) \leq C(f, D), \text{ for every } Q \in D,$$

where the constant  $C(f, D)$  depends only on  $f$  and  $D$ .

*Proof.* Without loss of generality, we may suppose that the point  $P$  is the origin of  $K^n$ . The hypothesis that the origin of  $K^n$  is an absolutely algebraically isolated singularity and the Hilbert Nullstellensatz imply that

$$\pi^{m_i} x_i^{t_i} = A_{i,0}(x) f(x) + \sum_{j=1}^n A_{i,j}(x) \frac{\partial f}{\partial x_j}(x), \quad (2.5)$$

for some  $m_i, t_i \in \mathbb{N}$  and some polynomials  $A(x)_{i,j} \in \mathcal{O}_K[x]$ , for each  $i = 0, 1, 2, \dots, n$ . Now, let  $Q = (q_1, \dots, q_n)$  be a point of  $D$ . Since  $D \cap A_r = \emptyset$ , there exists a coordinate  $j_0$  such that  $v(q_{j_0}) < r_{j_0}$ . From (2.5), with  $x = Q$  and  $i = j_0$ , we obtain

$$m_{j_0} + t_{j_0} r_{j_0} \geq m_{j_0} + t_{j_0} v(q_{j_0}) \geq L(Q, f).$$

Thus, it is sufficient to take  $C(f, D) \geq \max\{r_i + t_i m_i\}$ .  $\square$

The following result is a generalization of proposition 18 (cf. [13], sect. 18) of Néron.

**Proposition 2.3.** (Néron, [13], sect. 18, prop. 18) *Let  $f(x) \in \mathcal{O}_K[x]$  be a polynomial and  $P \in \mathcal{O}_K^n$  a point such that  $P \notin \text{Sing}_f(\mathcal{O}_K)$ , and  $\overline{P} \in \text{Sing}_{\overline{f}}(\mathbb{F}_q)$ . Then there exists a minimal non-negative integer  $\mu(f, P)$ , such that the polynomial*

$$f_P(x) = \pi^{-e_{\mu,P}} f(P + \pi^\mu x),$$

where  $e_{\mu,P}$  is the minimum order of  $\pi$  in the coefficients of  $f(P + \pi^\mu x)$ , satisfies

$$\overline{f_P(x)} = \alpha_0, \quad \alpha_0 \in \mathbb{F}_q^* \quad \text{or} \quad \overline{f_P(x)} = \sum \alpha_i x_i, \quad \alpha_i \in \mathbb{F}_q^*, \text{ for some } i = 1, 2, \dots, n.$$

Furthermore,  $\mu(f, P) \leq L(f, P) + 2$ .

*Proof.* Let  $P = (b_1, \dots, b_n) \in \mathcal{O}_K^n$  be a point such that  $\overline{P} \in \text{Sing}_{\overline{f}}(\mathbb{F}_q)$ . Since  $P \notin \text{Sing}_f(\mathcal{O}_K)$ , we have

$$f(x) = \alpha_0 + \sum_i \alpha_i (x_i - b_i) + (\text{degree} \geq 2),$$



where  $\alpha_i \equiv 0 \pmod{\pi}$ ,  $i = 0, 1, \dots, n$ . Thus

$$f(P + \pi x) = \pi \left( \alpha'_0 + \sum \alpha_i x_i + \pi(\text{degree} \geq 2) \right).$$

We consider two cases according to  $\alpha'_0 \not\equiv 0 \pmod{\pi}$  or not.

*Case 1* ( $\alpha'_0 \not\equiv 0 \pmod{\pi}$ ).

In this case, we have

$$f(P + \pi x) = \pi f_P(x),$$

where

$$f_P(x) = \alpha'_0 + \sum \alpha_i x_i + \pi(\text{degree} \geq 2).$$

Therefore,  $\overline{f_P(x)} = \overline{\alpha'_0} \in \mathbb{F}_q^*$ , and  $\mu(f, P) = 1 \leq L(f, P)$ .

*Case 2* ( $\alpha'_0 \equiv 0 \pmod{\pi}$ ).

In this case, we have

$$f(P + \pi x) = \pi^2 \left( \alpha''_0 + \sum_i \alpha'_i x_i + (\text{degree} \geq 2) \right) = \pi^{e_{\mu, P}} f_P(x),$$

where  $e_{\mu, P} \geq 2$ . Thus

$$f(P) = \pi^{e_{\mu, P}} f_P(0),$$

and

$$\frac{\partial f}{\partial x_i}(P) = \pi^{e_{\mu, P}-1} \frac{\partial f_P}{\partial x_i}(0), \quad i = 1, \dots, n.$$

Whence  $L(f_P, 0) \leq L(f, P) - 1$ . Thus after a finite number of dilatations, we obtain  $L(f_{P_1, \dots, P_k}, 0) = 0$ , ( where  $P_2, \dots, P_k$  are equal to the origin of  $K^n$ ), i.e.

$$f_{P_1, \dots, P_k}(x) = \alpha_0 + \sum \alpha_i x_i + (\text{degree} \geq 2),$$

where  $\alpha_0 \not\equiv 0 \pmod{\pi}$  or  $\alpha_i \not\equiv 0 \pmod{\pi}$ , for some  $i$ ,  $1 \leq i \leq n$ . If  $\alpha_0 \not\equiv 0 \pmod{\pi}$ , then after an additional dilatation, we obtain  $\overline{f_{P_1, \dots, P_k}(x)} = \overline{\alpha_0} \in \mathbb{F}_q^*$ , and  $\mu(f, P) \leq L(f, P) + 1$ . If  $\alpha_0 \equiv 0 \pmod{\pi}$ , then after an additional dilatation at the origin, we obtain

$$\overline{f_{P_1, \dots, P_k}(x)} = \sum \overline{\alpha_i} x_i,$$

where  $\overline{\alpha_i} \neq 0$  for some  $i$ , and  $\mu(f, P) \leq L(f, P) + 2$ .  $\square$

As a consequence of the two above results, we obtain the following lemma.

**Lemma 2.4.** *Let  $f(x) \in \mathcal{O}_K[x]$  be a polynomial such that the origin of  $K^n$  is an absolutely algebraically isolated singularity. Let  $A_r$  be a polydisc with  $r \in (\mathbb{N} \setminus \{0\})^n$ . Then there exists  $\gamma = \gamma(f, r) \in \mathbb{N}$ , such that the polynomial*

$$f_Q(x) = \pi^{-e_Q, \gamma} f(Q + \pi^\gamma x),$$

*satisfies the condition,  $\overline{f_Q(x)}$  is a non-zero constant or a linear polynomial without constant term, for all  $Q \in A_r^c$ .*

### 3. Rationality of Igusa's local zeta functions of semiquasihomogeneous polynomials

In this section, we prove the rationality of the Igusa local zeta function of semiquasihomogeneous polynomials.

**Lemma 3.1.** *Let  $D \subseteq \mathcal{O}_K^n$  be the preimage under the canonical homomorphism  $\mathcal{O}_K \rightarrow \mathcal{O}_K/\pi\mathcal{O}_K$  of a subset  $\overline{D} \subseteq \mathbb{F}_q^n$ . Let  $f(x) \in \mathcal{O}_K[x]$  be a polynomial such that the origin of  $K^n$  is an absolutely algebraically isolated singularity of  $V_f(K)$ . If  $D \cap A_r = \emptyset$ , for some  $r \in (\mathbb{N} \setminus \{0\})^n$ , then the integral  $Z(f, D, s) = \int_D |f|_K^s |dx|$  is a rational function of  $q^{-s}$ . More precisely,*

$$Z(f, D, s) = \frac{L(q^{-s}, D)}{1 - q^{-1}q^{-s}}. \quad (3.1)$$

Furthermore, the polynomial  $L(q^{-s}, D)$  can be effectively computed.

*Proof.* Applying the stationary phase formula  $m + 1$ -times, we obtain

$$\begin{aligned} Z(f, D, s) &= \sum_{k=0}^m q^{-kn} \left( \sum_{P_1, P_2, \dots, P_k} \nu(\overline{f}_{P_1, \dots, P_k}) q^{-E(P_1, \dots, P_k)s} \right) + \\ &\quad \frac{(1 - q^{-1})q^{-s}}{(1 - q^{-1}q^{-s})} \sum_{k=0}^m q^{-kn} \left( \sum_{P_1, P_2, \dots, P_k} \sigma(\overline{f}_{P_1, \dots, P_k}) q^{-E(P_1, \dots, P_k)s} \right) \\ &+ \sum_{P_1, \dots, P_m} \sum_{P_{m+1} \in S(f_{P_1, \dots, P_m})} q^{-(m+1)n - E(P_1, \dots, P_m)s} \int_{\mathcal{O}_K^n} |f_{P_1, \dots, P_m}(x)|_K^s |dx|. \end{aligned} \quad (3.2)$$

On the other hand, we have

$$f(P_1 + P_2\pi + \dots + P_m\pi^{m+1} + \pi^{m+2}x) = \pi^{E(P_1, \dots, P_m)} f_{P_1, \dots, P_m}(x). \quad (3.3)$$

Since  $P_1 + P_2\pi + \dots + P_m\pi^{m+1} \in A_r^c$ , lemma 2.4 and (3.3) imply that if  $m+2 \geq \gamma(f, r)$ , then the  $\mathbb{F}_q$ -hypersurface defined by  $\overline{f_{P_1, \dots, P_m}(x)}$  is smooth or empty, whence the corresponding integral in (3.2) can be computed using the stationary phase formula. Therefore the integral  $Z(f, D, s)$  can be computed applying  $\gamma(f, r)$  times the stationary phase formula.  $\square$

**Lemma 3.2.** *Let  $f(x) \in \mathcal{O}_K[x]$  be a polynomial such that the origin of  $K^n$  is an absolutely algebraically isolated singularity of  $V_f(K)$ . Then the integral  $Z(f, A_r^c, s) = \int_{A_r^c} |f|^s |dx|$  is a rational function of  $q^{-s}$ . More precisely,*

$$Z(f, A_r^c, s) = \frac{L(q^{-s})}{1 - q^{-1}q^{-s}}. \quad (3.4)$$

Furthermore, the polynomial  $L(q^{-s})$  can be computed effectively.

*Proof.* We introduce a family  $\mathcal{L}$  of sets defined as follows. For each subset  $B$  of  $\{1, \dots, n\}$  and each  $a = (a_1, \dots, a_n) \in \mathbb{N}^n$  satisfying  $0 \leq a_i < r_i$  if  $i \in B$ , we define

$$D(B, a) := \{x \in A_r^c \mid v(x_i) = a_i \text{ if } i \in B\}, \text{ if } B \neq \emptyset,$$

$$D(B, a) := \emptyset, \text{ if } B = \emptyset. \quad (3.5)$$

The family  $\mathcal{L}$  is closed under intersections, and its union is  $A_r^c$ . We denote by  $J$  the set of indices  $\{(B, a)\}$  and by  $\mathcal{P}(J)_i$  the family of subsets of  $J$  with  $i$  elements.

Whence

$$Z(f, A_r^c, s) = \sum_{i=1}^{\text{Card}\{J\}} (-1)^{i-1} \sum_{T \in \mathcal{P}(J)_i} \int_{D(T)} |f|^s |dx|, \quad (3.6)$$

where  $D(T) := \bigcap_{(B, a) \in T} D(B, a)$ . From (3.6) and the fact that the family  $\mathcal{L}$  is closed under intersections, it follows that in order to prove the theorem, it is sufficient to prove that any integral of type

$$\int_{D(B, a)} |f|^s |dx|, \quad B \neq \emptyset \quad (3.7)$$

is a rational function of the form  $L(q^{-s}, D(B, a))/(1 - q^{-1}q^{-s})$ , where the numerator polynomial is effectively computable. For that, we make the following change of variables in (3.7),

$$x = \psi_{(B, a)}(y), \text{ where } x_i = \begin{cases} \pi^{a_i} y_i & \text{if } i \in B \\ y_i & \text{if } i \notin B. \end{cases} \quad (3.8)$$

we obtain

$$\int_{D(B, a)} |f|^s |dx| = q^{-e_{(B, a)}s - d_{(B, a)}} \int_{D'(B, a)} |f_B|^s |dy|, \quad (3.9)$$

where  $d_{(B, a)} = \sum_{i \in B} a_i$ ,  $f_B(y)$  is the dilatation of  $f$  induced by (3.8) and

$$D'(B, a) = \prod_{i=1}^n R_i,$$

where  $R_i = \mathcal{O}_K$  if  $i \notin B$  and  $R_i = \mathcal{O}_K^*$  if  $i \in B$ .

On the other hand,  $\phi(y)$  defines a  $K$ -isomorphism of  $K^n \rightarrow K^n$ , thus the  $K$ -singular locus of  $V_f$  is mapped bijectively on the  $K$ -singular locus of  $V_{f_B}$ . Therefore, the polynomial  $f_B$  and the set  $D'(B, a)$ ,  $B \neq \emptyset$ , satisfy the conditions of lemma 3.1. Thus the integral  $\int_{D'(B, a)} |f_B|^s |dx|$  is a rational function of  $q^{-s}$  and its numerator can be computed effectively.  $\square$

**Proposition 3.3.** *Let  $F(x) = f(x) + \pi^m g(x) \in \mathcal{O}_K[x]$  be a semiquasihomogeneous polynomial,  $f(x)$  is its quasihomogeneous part. Let  $D \subseteq \mathcal{O}_K^n$  be the preimage under the canonical homomorphism  $\mathcal{O}_K \rightarrow \mathcal{O}_K/\pi\mathcal{O}_K$  of a subset  $\overline{D} \subseteq \mathbb{F}_q^n$ , and  $A_r$  a polydisc, such that  $D \cap A_r = \emptyset$ ,  $r \in (\mathbb{N} \setminus \{0\})^n$ . There exists  $\alpha(f, D)$ , effectively computable, such that if  $m \geq \alpha(f, D)$  then*

$$Z(F, D, s) = Z(f, D, s).$$

*Proof.* By virtue of lemma 2.4, there exists a  $\gamma_0$  such that the reduction modulo  $\pi$  of the polynomial

$$F_{P_1, \dots, P_n}(x) = \pi^{-E_F(P_1, \dots, P_n)} F(P_1 + P_2\pi + \dots + P_n\pi^n + \pi^{n+1}x) \quad (3.10)$$

is a non-zero constant or a linear polynomial for every  $n \geq \gamma_0$  and any  $P_1 + P_2\pi + \dots + P_n\pi^n \in A_r^c$ . In addition, we also have that

$$F(P_1 + P_2\pi + \dots + P_n\pi^n + \pi^{n+1}x) = \pi^{E_f(P_1, \dots, P_n)} f_{P_1, \dots, P_n}(x)$$

$$+\pi^{E_g(P_1, \dots, P_n)+m} g_{P_1, \dots, P_n}(x). \quad (3.11)$$

We choose

$$\alpha(f, D) := \text{Max} \{E_f(P_1, \dots, P_{\gamma_0})\}, \quad (3.12)$$

where  $P_{\gamma_0}$  runs through the  $\gamma_0$  generation of descendants of  $S(F, D)$ . Now, If  $m > \alpha(f, D)$ , we have

$$\begin{aligned} \overline{F_{P_1, P_2, \dots, P_k}(x)} &= \overline{f_{P_1, P_2, \dots, P_k}(x)}, \\ E_{P_1, P_2, \dots, P_k}(F) &= E_{P_1, P_2, \dots, P_k}(f), \quad 1 \leq k \leq \gamma_0. \end{aligned} \quad (3.13)$$

Expanding  $Z(F, D, s)$  and  $Z(f, D, s)$  as in (3.2) and using (3.10) and (3.13), we conclude that  $Z(F, D, s) = Z(f, D, s)$ .

□

**Lemma 3.4.** *Let  $F(x) = f(x) + \pi^m g(x) \in \mathcal{O}_K[x]$  be a semiquasihomogeneous polynomial,  $f(x)$  is its quasihomogeneous part. Let  $A_r$ , be a polydisc, with  $r \in (\mathbb{N} \setminus \{0\})^n$ . There exists  $\alpha(f, r)$ , effectively computable, such that if  $m \geq \alpha(f, r)$  then*

$$Z(F, A_r^c, s) = Z(f, A_r^c, s).$$

*Proof.* By (3.6), it is sufficient to prove the lemma for the integrals of type (3.7). We choose  $\alpha(f, r)$  satisfying

$$\alpha(f, r) \geq \text{Max}_{(B, a)} \{e_{f, (B, a)}\}.$$

With the above condition, we have that (see (3.9)) it is sufficient to prove that

$$\int_{D(B, a)'} |F_B|^s |dx| = \int_{D(B, a)'} |f_B|^s |dx|. \quad (3.14)$$

The result follows from (3.14) and proposition 3.3. Finally, we observe that  $\alpha(f, r)$  is given by

$$\alpha(f, r) = \text{Max}_{(B, a)} \{e_{f, (B, a)} + \alpha(f, D'(B, a))\}.$$

□

**Theorem 3.5.** *Let  $F(x) \in K[x]$  be an a semiquasihomogeneous polynomial whose quasihomogeneous part  $f(x)$  has weight  $d$  and exponents  $\alpha_1, \dots, \alpha_n$ . Then Igusa's local zeta function of  $F(x)$  is a rational function of  $q^{-s}$ . More precisely,*

$$Z(F, s) = \frac{L(q^{-s})}{(1 - q^{-1}q^{-s})(1 - q^{-|\alpha|}q^{-ds})}. \quad (3.15)$$

where  $\alpha = (\alpha_1, \dots, \alpha_n)$ . Furthermore, the polynomial  $L(q^{-s})$  can be computed effectively.

*Proof.* By decomposing  $\mathcal{O}_K^n$  as the disjoint union of  $A_\alpha$  and  $A_\alpha^c$  and using the fact that  $F(x)$  is a semiquasihomogeneous polynomial, we obtain

$$Z(F, s) = \int_{A_\alpha} |F|_K^s |dx| + \int_{A_\alpha^c} |F|_K^s |dx| = q^{-|\alpha|-ds} \int_{\mathcal{O}_K^n} |F_1|_K^s |dx| + \int_{A_\alpha^c} |F|_K^s |dx|, \quad (3.16)$$

where  $F_1(x) = f(x) + \pi^{m_1} H_1(x)$ , with  $m_1 \geq 1$ . Now, Iterating formula (3.16)  $m$ -times, we obtain:

$$Z(F, s) = Z(F, A_\alpha^c, s) + \sum_{k=1}^m q^{k(-|\alpha|-ds)} Z(F_k, A_\alpha^c, s) + q^{(m+1)(-|\alpha|-ds)} Z(F_{m+1}, s), \quad (3.17)$$

where  $F_k(x) = f(x) + \pi^{m_k} H_k(x)$ , with  $m_k \rightarrow \infty$ . By lemma 3.4, there exists  $\gamma_0 = \gamma_0(f, \alpha)$ , effectively computable, such that  $Z(F_k, A_\alpha^c, s) = Z(f, A_\alpha^c, s)$  if  $m \geq \gamma_0$ . Thus from (3.17), we have

$$Z(F, s) = Z(F, A_\alpha^c, s) + \sum_{k=1}^{m_0-1} q^{k(-|\alpha|-ds)} Z(F_k, A_\alpha^c, s) + Z(f, A_\alpha^c, s) q^{(m_0+1)(-|\alpha|-ds)} \frac{1}{1 - q^{-|\alpha|-ds}}. \quad (3.18)$$

By lemma 3.2 the integrals  $Z(f, A_\alpha^c, s)$  and  $Z(F_k, A_\alpha^c, s)$  are rational functions of  $q^{-s}$  of the form  $\frac{L(q^{-s})}{1 - q^{-|\alpha|-ds}}$ , where the polynomial numerator can be effectively computed.  $\square$

We observe that if  $f$  is a quasihomogeneous polynomial, its local zeta function is given by  $Z(f, s) = \frac{Z(f, A_\alpha^c, s)}{1 - q^{-|\alpha|-ds}}$ . The integral  $Z(f, A_\alpha^c, s)$  can be computed using lemma 3.2.

As a consequence of theorem 3.5, we obtain the following three corollaries.

**Corollary 3.6.** *Let  $K$  be a global field and let  $F(x) \in K[x]$  be a semiquasihomogeneous polynomial whose quasihomogeneous part  $f(x)$  has weight  $d$  and exponents  $\alpha_1, \dots, \alpha_n$ . Then for every non-archimedean valuation  $v$  of  $K$ , Igusa's local zeta function of  $F(x)$  on the completion  $K_v$  of  $K$  is a rational function of form (3.15). If  $K$  is a number field and  $F(x)$  is non-degenerate for its Newton's diagram, then the real parts of the poles of the zeta function  $Z(F, s)$  are roots of the Bernstein polynomial of  $F(x)$ .*

*Proof.* Since  $F(x)$  has an absolutely algebraically isolated singularity at the origin of  $K^n$ , the Hilbert Nullstellensatz implies that for all valuations  $v$  of  $K$ , the origin of

$K_v^n$  is an absolutely algebraically isolated singularity of  $V_f(K_v)$ . Thus by the proof of theorem 3.5, the denominator of the the local zeta function  $Z(f, s)$  on  $K_v$  is equal to  $(1 - q^{-1}q^{-s})(1 - q^{-|\alpha|}q^{-ds})$ . Thus the real parts of the poles of  $Z(F, s)$  are among the values  $-1, -|\alpha|/d$ . If  $K$  is a number field, and  $F(x)$  is non-degenerate for its Newton's diagram, theorem C.2.2.3 of [2] implies that  $-1$  and  $-|\alpha|/d$  are roots of the Bernstein polynomial of  $F(x)$ .  $\square$

The following corollary gives a bound for the number of solutions of a congruence attached to a semiquasihomogeneous polynomial with coefficients in a ring of integers of a global field. This corollary follows directly from the relation existing between the Igusa local zeta function and the Poincaré series  $P(t)$  (see introduction) and corollary 3.6.

**Corollary 3.7.** *Let  $K$  be a global field and  $\mathcal{O}_K$  its ring of integers. Let  $F(x) \in \mathcal{O}_K[x]$  be a semiquasihomogeneous polynomial whose quasihomogeneous part  $f(x)$  has weight  $d$  and exponents  $\alpha_1, \dots, \alpha_n$ . Then for every non-archimedean valuation  $v$  of  $K$ , the number of solutions  $N_j(F, v)$  of the congruence*

$$F(x) \equiv 0 \pmod{\pi^j \mathcal{O}_{K_v}},$$

where  $\mathcal{O}_{K_v}$  is the ring of integers of the completion  $K_v$ , satisfies

$$\limsup_{j \rightarrow \infty} N_j(F, v)^{1/j} \leq \begin{cases} q^{n-|\alpha|/d} & \text{if } |\alpha|/d \leq 1, \\ q^{n-1} & \text{if } |\alpha|/d > 1. \end{cases}$$

The following corollary follows directly from theorem 3.5. We use the notation established in the introduction.

**Corollary 3.8.**

*Let  $f(x, y) \in K[x, y]$  be an absolutely analytically irreducible polynomial, such that the origin of  $K^2$  is an absolutely algebraically isolated singularity of the plane curve  $V_{f_d}$ . Then the Igusa local zeta function  $Z(f, s)$  is a rational function of  $q^{-s}$  of form (3.15).*

**Example 3.9.** In this example we compute the local zeta function for a polynomial of type  $f(x, y) = \alpha x^n + \beta y^m$ ,  $\alpha, \beta \in \mathcal{O}_K$ , where  $n, m > 1$  are relatively prime. Suppose that the characteristic of  $K$  does not divide both  $n, m$ . Furthermore, without loss of generality, we may suppose that  $\alpha \in \mathcal{O}_K^*$ . In the case of characteristic zero, the Poincaré series  $P(t)$  associated to this type of polynomials were explicitly computed by Goldman (cf. [6], thm. 1).

Using the observation made after the proof of theorem 3.5, we have

$$Z(f, s) = \frac{1}{1 - q^{-(n+m)-mns}} \int_{A^c} |f|_K^s |dx dy|, \quad (3.19)$$

where  $A = \{(x, y) \in \mathcal{O}_K^2 \mid v(x) \geq m, v(y) \geq n\}$ . The complement  $A^c$  of  $A$  is the disjoint union of the following three sets:

$$D_1 = \{(x, y) \in \mathcal{O}_K^2 \mid v(x) < m, v(y) \geq n\},$$

$$D_2 = \{(x, y) \in \mathcal{O}_K^2 \mid v(x) < m, v(y) < n\}.$$

$$D_3 = \{(x, y) \in \mathcal{O}_K^2 \mid v(x) \geq m, v(y) < n\}.$$

Thus from (3.19), we obtain

$$Z(f, s) = \frac{1}{1 - q^{-(n+m)-mns}} \{Z(f, D_1, s) + Z(f, D_2, s) + Z(f, D_3, s)\}. \quad (3.20)$$

Next, we compute the integrals  $Z(f, D_1, s)$ ,  $Z(f, D_2, s)$ ,  $Z(f, D_3, s)$ .

#### Computation of $Z(f, D_1, s)$

First, we observe that

$$|f(x, y)| = |\alpha x^n + \beta y^m| = |x^n|, \quad x, y \in D_1.$$

Therefore

$$Z(f, D_1, s) = \int_{D_1} |f|_K^s |dx dy| = \sum_{k=0}^{m-1} \int_{\{(x, y) \in D_1 \mid v(x)=k, v(y) \geq n\}} |x|_K^{ns} |dx dy|.$$

Thus

$$Z(f, D_1, s) = (1 - q^{-1}) q^{-n} \sum_{k=0}^{m-1} q^{-kns-k} \quad (3.21)$$

#### Computation of $Z(f, D_2, s)$

We set  $L(i, j) := jm - in + v(\beta)$ . The set  $D_2$  can be decomposed as the union of three disjoint subsets  $D_{2,1}, D_{2,2}, D_{2,3}$ , as follows :

$$D_{2,1} := \{(x, y) \in D_2 \mid L(v(x), v(y)) > 0\},$$

$$D_{2,2} := \{(x, y) \in D_2 \mid L(v(x), v(y)) < 0\},$$

$$D_{2,3} := \{(x, y) \in D_2 \mid L(v(x), v(y)) = 0\}.$$



Thus  $Z(f, D_2, s) = Z(f, D_{2,1}, s) + Z(f, D_{2,2}, s) + Z(f, D_{2,3}, s)$ , where

$$Z(f, D_{2,1}, s) = (1 - q^{-1})^2 \sum_{i,j} q^{-i-j-nis}, \quad (3.22)$$

where  $i, j$  satisfy  $L(i, j) > 0$  and  $0 \leq i < m$ ,  $0 \leq j < n$ ,

$$Z(f, D_{2,2}, s) = (1 - q^{-1})^2 \sum_{i,j} q^{-i-j-(v(\beta)+mj)s}, \quad (3.23)$$

where  $i, j$  satisfy  $L(i, j) < 0$  and  $0 \leq i < m$ ,  $0 \leq j < n$ , and

$$Z(f, D_{2,3}, s) = \sum_{i,j} q^{-i-j-nis} \int_{\mathcal{O}_K^{\times 2}} |\alpha x^n + \mu y^m|_K^s |dx dy|, \quad (3.24)$$

where  $\beta = \pi^{v(\beta)}\mu$ ,  $\mu \in \mathcal{O}_K^*$ ,  $i, j$  satisfy  $L(i, j) = 0$  and  $0 \leq i < m$ ,  $0 \leq j < n$ . Using the stationary phase formula, we compute the integral in the right side of (3.24), thus

$$Z(f, D_{2,3}, s) = \left( \nu(\bar{f}) + \frac{\sigma(\bar{f})(1 - q^{-1})q^{-s}}{1 - q^{-1-s}} \right) \sum_{i,j} q^{-i-j-nis}.$$

We denote by  $[x]$  the integer part of a real number  $x$ . We set  $v(\beta) = gn + r$ ,  $0 \leq r < n$ .

**Computation of  $Z(f, D_3, s)$**

We set

$$D_{3,1} := \{(x, y) \in \mathcal{O}_K^2 \mid v(x) \geq m + [\frac{v(\beta)}{n}] + r, \ v(y) < n\},$$

$$D_{3,2} := \{(x, y) \in \mathcal{O}_K^2 \mid m \leq v(x) \leq m + [\frac{v(\beta)}{n}] + r - 1, \ v(y) < n\}.$$

Then  $D_3 = D_{3,1} \cup D_{3,2}$  (disjoint union), and  $Z(f, D_3, s) = Z(f, D_{3,1}, s) + Z(f, D_{3,2}, s)$ . To compute  $Z(f, D_{3,1}, s)$ , we observe that

$$|f(x, y)| = |\alpha x^n + \beta y^m| = |\beta y^m| \quad x, y \in D_{3,1}.$$

Thus

$$Z(f, D_{3,1}, s) = \int_{D_{3,1}} |f(x, y)|_K^s |dx dy| = (1 - q^{-1}) q^{-(m + [\frac{v(\beta)}{n}] + r)s} \sum_{k=0}^{n-1} q^{-(v(\beta) + mk)s - k}. \quad (3.25)$$

The set  $D_{3,2}$  can be decomposed as the union of three disjoint subsets  $D_{3,2,1}$ ,  $D_{3,2,2}$ ,  $D_{3,2,3}$ , as follows :

$$D_{3,2,1} := \{(x, y) \in D_{3,2} \mid L(v(x), v(y)) > 0\},$$

$$D_{3,2,2} := \{(x, y) \in D_{3,2} \mid L(v(x), v(y)) < 0\},$$

$$D_{3,2,3} := \{(x, y) \in D_{3,2} \mid L(v(x), v(y)) = 0\}.$$

Thus  $Z(f, D_{3,2}, s) = Z(f, D_{3,2,1}, s) + Z(f, D_{3,2,2}, s) + Z(f, D_{3,2,3}, s)$ , and

$$Z(f, D_{3,2,1}, s) = (1 - q^{-1})^2 \sum_{i,j} q^{-i-j-nis}, \quad (3.26)$$

where  $i, j$  satisfy  $L(i, j) > 0$  and  $m \leq i < m + [\frac{v(\beta)}{n}] + r$ ,  $0 \leq j < n$ ,

$$Z(f, D_{3,2,2}, s) = (1 - q^{-1})^2 \sum_{i,j} q^{-i-j-(v(\beta)+mj)s}, \quad (3.27)$$

where  $i, j$  satisfy  $L(i, j) < 0$  and  $m \leq i < m + [\frac{v(\beta)}{n}] + r$ ,  $0 \leq j < n$ , and

$$Z(f, D_{3,2,3}, s) = \left( \nu(\bar{f}) + \frac{\sigma(\bar{f})(1 - q^{-1})q^{-s}}{1 - q^{-1-s}} \right) \sum_{i,j} q^{-i-j-nis},$$

where  $i, j$  satisfy  $L(i, j) = 0$  and  $m \leq i < m + [\frac{v(\beta)}{n}] + r$ ,  $0 \leq j < n$ .

**Example 3.10.** A polynomial of the form  $f(x) = \sum_i \alpha_i x_i^{n_i}$ ,  $\alpha_i \in \mathcal{O}_K$  is called a diagonal polynomial. We set  $d := l.c.m\{n_i\}$ , and  $\alpha_i := d/n_i$ ,  $i = 1, \dots, n$ . If the characteristic of  $K$  does not divide any  $n_i$ , then the diagonal polynomials are quasihomogeneous polynomials with exponents  $\alpha_i := n_i/d$ ,  $i = 1, \dots, n$  and weight  $d$ . Thus the local zeta function of a diagonal polynomial is a rational function of form (3.13). Wang and others have studied the Poincaré series  $P(t)$  associated to this class of polynomials (cf [14], thm. 1).

### Acknowledgements.

The author wishes to thank to the following institutions for their support: Universidad Autónoma de Bucaramanga, Academia Colombiana de Ciencias Exactas, Físicas y Naturales and COLCIENCIAS. The author also thanks IMPA for their support and hospitality during the summer of 1997, when part of this work was done. The author also wishes to thank to referee for his or her useful comments which lead to an improvement of this work.

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UNIVERSIDAD AUTÓNOMA DE BUCARAMANGA, LABORATORIO DE COMPUTO ESPECIALIZADO, A.A.  
1642, BUCARAMANGA, COLOMBIA

*E-mail address:* wzuniga@bumanga.unab.edu.co